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## SUDDEN BENDING OF CRACKED LAMINATES

BY

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16. Abstract  <p>A dynamic approximate laminated plate theory is developed with emphases placed on obtaining effective solution for the crack configuration where the <math>1/\sqrt{r}</math> stress singularity and the condition of plane strain are preserved. The radial distance <math>r</math> is measured from the crack edge. The results obtained show that the crack moment intensity tends to decrease as the crack length to laminate plate thickness is increased. Hence, a laminated plate has the desirable feature of stabilizing a through crack as it increases its length at constant load. Also, the level of the average load intensity transmitted to a through crack can be reduced by making the inner layers to be stiffer than the outer layers. The present theory, although approximate, is useful for analyzing laminate failure to crack propagation under dynamic load conditions.</p>					
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## FOREWORD

The research results in this report on the sudden bending of a laminated plate containing a through crack represent a portion of the work performed for the NASA - Lewis Research Center in Cleveland, Ohio for the period February 13, 1979 through February 12, 1980 under Grant NSG 3179 with the Institute of Fracture and Solid Mechanics at Lehigh University. The Principal Investigator of the project is Professor George C. Sih. The co-author, Dr. E. P. Chen, was a faculty member at Lehigh University and is now employed by the Sandia Laboratory in New Mexico. The encouragement and helpful comments made by Dr. Christos C. Chamis, the NASA Project Manager, are gratefully acknowledged.

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# LIST OF SYMBOLS

$a$	- half of the crack length
$B^{(1)}, B^{(2)}, B^{(3)}$	- unknowns in integrals, function of $(s, p)$
$Br$	- Bromwich contour in the complex $p$ -plane
$C$	- unknown in dual integral equation, function of $(s, p)$
$c_{21}$	- shear wave speed for medium 1
$D_0, D_1, D_2$	- flexural rigidities of the layers in the laminate
$f^*(p)$	- Laplace transform of $f(t)$
$G$	- known function of $(s, p)$ in the dual integral equation
$h$	- laminate thickness
$H^*$	- function related to displacement
$H(t)$	- Heaviside unit step function
$J_0$	- Bessel function of order zero
$\bar{K}$	- kinetic energy
$K_1(t)$	- moment intensity factor
$L(\xi, n, p)$	- kernel in Fredholm integral equation
$M_0$	- magnitude of applied moment
$M_x, M_y, H_{xy}$	- moments per unit length defined in the $xy$ coordinate system
$M_n, H_{ns}, Q_n$	- moments and shear per unit length defined in the normal and tangential directions
$p$	- Laplace transform variable
$q_j$	- lateral loadings on laminate with $j = 1, 2$
$q$	- defined as $q_1 - q_2$
$Q_x, Q_y$	- shear force per unit length
$r, \theta$	- crack edge polar coordinates
$R, S$	- parameters in the laminate plate theory

$t$	- time
$\bar{T}, \bar{V}$	- energy quantities at time $t$
$\bar{T}_0, \bar{V}_0$	- energy quantities at time $t=0$
$u_x, v_y, w_z$	- displacement components in the $(x,y,z)$ coordinate system
$w, w_j$	- displacement functions with $j = 1, 2$
$x, y, z$	- rectangular coordinates
$\alpha_j, \beta_j$	- parameters related to $\alpha$ and $\beta$ with $j = 1, 2$
$\gamma_j$	- exponents for transform of solution with $j = 1, 2, 3$
$\Gamma_x, \Gamma_y, \dots, \Gamma_{yz}$	- equivalent strains
$\delta_0, \bar{\delta}_0$	- parameters in plate theory, function of $p$
$\epsilon_x, \epsilon_y, \dots, \gamma_{yz}$	- strain components
$\kappa$	- shear correction parameter in plate theory
$\mu_0$	- equivalent shear modulus
$\mu_j$	- shear modulus with $j = 1, 2$
$\nu_0$	- equivalent Poisson's ratio
$\nu_j$	- Poisson's ratio with $j = 1, 2$
$\xi, \eta$	- variables of integration
$\rho_0, \bar{\rho}$	- function of $\rho_1$ and $\rho_2$
$\rho_j$	- mass density for medium $j$
$\rho(z)$	- mass density as a function of $z$
$\sigma_x, \sigma_y, \dots, \tau_{yz}$	- stress components
$\phi^*$	- displacement function in Laplace transform plane
$\psi_x, \psi_y$	- displacement functions in the $xy$ coordinate system
$\psi_n, \psi_s$	- displacement functions in the $ns$ coordinate system
$\Psi^*(\xi, p)$	- unknown in Fredholm integral equation



- $\Psi^*(1,p)$  - value of  $\Psi^*(\xi,p)$  evaluated at  $\xi=1$
- $\omega$  - frequency parameter
- $\nabla^2$  - Laplacian operator

# SUDDEN BENDING OF A CRACKED LAMINATE

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## ABSTRACT

A number of laminated plate theories have been developed in recent times to analyze the static and dynamic response of composite laminates with or without the presence of stress concentrators such as holes, cracks, etc. Many of the theories tend to quickly become intractable when considering the determination of the state of affairs near the singular crack edges that are present in the laminate, particularly if the loading is time dependent. Additional uncertainties arise due to the lack of information on the mechanical properties of the interface through which load transfer takes place between the adjacent layers. This paper focuses attention on the intensification of stresses near a through crack in the laminate that suddenly undergoes bending. A dynamic plate theory is developed to include many of the essential features of the problem such as material nonhomogeneity in the thickness direction, realistic crack edge stress singularity and distribution while the parameter dependence of various significant quantities is also assessed. Of particular interest is the variation of the dynamic stress intensity factor with time. Numerical results for different

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<sup>\*</sup>Dr. E. P. Chen was on the faculty at Lehigh University.

geometric and material constants are displayed graphically to show how they can affect the transfer of load to the vicinity of a through crack in the laminate that undergoes sudden bending.

### INTRODUCTION

The damage of laminated composite materials is, to say the least, very complex since it involves various modes of failure such as fiber breaking, matrix cracking, interface delamination, etc. Analytical modeling would be beyond approach if all these failure modes were to be accounted for. The spirit of fracture mechanics is to assume that a critical single flaw or damage zone exists and can lead to instability in terms of load applied to the laminate. Damage accumulated in the composite other than the dominant flaw may often be simulated by changing some of the mechanical properties of the composite which are usually the stiffness of the constituents. Although not all laminates can be identified with a single characteristic damage state, the single-flaw fracture mechanics approach will be taken in this analysis in order that a sensitivity study on the physical parameters affecting laminate fracture can be made possible. One of the main objectives of this investigation is to come forth with a feasible dynamic theory of the laminate plate for analyzing composite failure due to crack propagation.

As a consequence of increased use of laminate composites in aircraft and other high speed vehicles, the analysis of the fracture behavior of layered composites has attracted the attention of a considerable number of investigators [1,2]. A variety of diverse approaches has been proposed to analyze laminate failure and a collection of papers on this subject can be found in [3]. The role with which the interfaces play in transferring the load from one layer to

the next in the laminate was emphasized. Because of the difference in the material properties of the adjacent layers, the stresses across the interface experience steep gradients. Only recently, a comprehensive study was made on how the conditions in the interface can influence composite failure [4]. Even though the interface may be relatively thin when compared with other dimensions of the composite, the resulting stresses can be sensitive to the material properties of the interface depending on the loading conditions. There exists no theory at the present which can relate the strength of a composite structure to the conditions in the interface. This aspect of the problem is emphasized in this report.

The aforementioned difficulties become even more overwhelming when the loading is time dependent. There is no need to emphasize the virtue for constructing approximate dynamic theories for laminate composites, particularly for handling crack problems. In the case of bending loads, it is essential that the three physical boundary conditions of bending moment, twisting moment and transverse shear stress be satisfied individually on the crack edge. Such a theory has been developed by Mindlin [5] for a single layered plate made of isotropic and homogeneous material and applied to solve a number of crack problems [6]. An equally effective theory is described herein for the dynamic bending of laminate plates. Each layer of the laminate assumes different elastic properties and is attached to the next layer with continuous strains across the interface. The problem of a through crack in a balanced symmetric laminate is solved for a moment applied suddenly on the crack surface. Not only are the qualitative features of the three-dimensional stress distribution preserved in the vicinity of the crack front, but, perhaps more significantly, the dynamic stress intensity

factor, which is a quantitative measure of the load transmitted to the crack, is determined in terms of the significant material and geometric parameters such that an effective study on laminate fracture can be made.

#### DYNAMIC THEORY OF LAMINATED PLATE

Without loss in generality, a four layered composite plate will be considered as shown in Figure 1. The two middle layers are made of a material with shear modulus  $\mu_1$ , Poisson's ratio  $\nu_1$  and mass density  $\rho_1$  while the two outer layers have the properties  $\mu_2$ ,  $\nu_2$  and  $\rho_2$ . A set of rectangular Cartesian coordinates  $x$ ,  $y$  and  $z$  are attached to the mid-plane of the laminate such that the layer properties are symmetric with respect to the  $xy$ -plane with  $z$  being the thickness coordinate. The total height of the laminate is  $h$  with each layer having the same thickness  $h/4$ . The outer edges of the laminate are sufficiently far away from the crack so that their influences can be neglected.

*Basic assumptions and relations.* The layers of the laminate in the thickness possess different material properties  $\mu_j$ ,  $\nu_j$  and  $\rho_j$  ( $j = 1, 2$ ) such that  $(\mu_1, \nu_1, \rho_1)$  prevails in the range  $0 < |z| < h/4$  and  $(\mu_2, \nu_2, \rho_2)$  applies to  $h/4 < |z| < h/2$ . The surfaces of the laminate are free from tangential tractions

$$\tau_{xz} = \tau_{yz} = 0 \text{ for } z = \pm h/2 \quad (1)$$

but may be subjected to normal pressures  $q_1$  and  $q_2$  as follows:

$$\begin{aligned} & -q_1(x,y,t) \text{ for } z = h/2 \\ \sigma_z = & \\ & -q_2(x,y,t) \text{ for } z = -h/2 \end{aligned} \quad (2)$$

In the sequel, the notation

$$q(x,y,t) = q_2(x,y,t) - q_1(x,y,t) \quad (3)$$

will be used. In plate theory, it is more convenient to work with the moments  $M_x$ ,  $M_y$ ,  $H_{xy}$  and shearing forces  $Q_x$ ,  $Q_y$  defined in the usual manner as

$$(M_x, M_y, H_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) z dz \quad (4)$$

$$(Q_x, Q_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) dz$$

From the stress and strain relations and equations (4), the expressions

$$\begin{aligned} M_x &= D_1[(r_x)_1 + \nu_1(r_y)_1] + D_2[(r_x)_2 + \nu_2(r_y)_2] \\ M_y &= D_1[(r_y)_1 + \nu_1(r_x)_1] + D_2[(r_y)_2 + \nu_2(r_x)_2] \\ H_{xy} &= \left(\frac{1-\nu_1}{2}\right) D_1(r_{xy})_1 + \left(\frac{1-\nu_2}{2}\right) D_2(r_{xy})_2 \end{aligned} \quad (5)$$

and

$$\begin{aligned} Q_x &= \frac{\kappa^2}{2} h[\mu_1(r_{xz})_1 + \mu_2(r_{xz})_2] \\ Q_y &= \frac{\kappa^2}{2} h[\mu_1(r_{yz})_1 + \mu_2(r_{yz})_2] \end{aligned} \quad (6)$$

are developed provided that the quantities  $(r_x)_j, (r_y)_j, \dots, (r_{yz})_j$  ( $j = 1, 2$ ) stand for

$$\begin{aligned} [(r_x)_1, (r_y)_1, (r_{xy})_1] &= \frac{96}{h^3} \int_{-h/4}^{h/4} (\epsilon_x, \epsilon_y, \gamma_{xy}) dz \\ [(r_x)_2, (r_y)_2, (r_{xy})_2] &= \frac{96}{7h^3} \left[ \int_{-h/2}^{-h/4} (\epsilon_x, \epsilon_y, \gamma_{xy}) dz \right. \\ &\quad \left. + \int_{h/4}^{h/2} (\epsilon_x, \epsilon_y, \gamma_{xy}) dz \right] \end{aligned} \quad (7)$$

$$\begin{aligned} [(r_{xz})_1, (r_{yz})_1] &= \frac{2}{h} \int_{-h/4}^{h/4} (\gamma_{xz}, \gamma_{yz}) dz \\ [(r_{xz})_2, (r_{yz})_2] &= \frac{2}{h} \left[ \int_{-h/2}^{-h/4} (\gamma_{xz}, \gamma_{yz}) dz + \int_{h/4}^{h/2} (\gamma_{xz}, \gamma_{yz}) dz \right] \end{aligned}$$

In equations (5),  $D_1$  and  $D_2$  are the flexural rigidities of the layers given by

$$D_1 = \frac{\mu_1 h^3}{48(1-\nu_1)}, \quad D_2 = \frac{7\mu_2 h^3}{48(1-\nu_2)} \quad (8)$$

The constant  $\kappa$  in equation (6) accounts for the thickness-shear motion of the plate and takes the value of  $\pi/\sqrt{12}$  as given in [5].

Now, let the displacements be continuous through the interfaces by letting

$$u_x = z\psi_x(x, y, t), \quad v_y = z\psi_y(x, y, t), \quad w_z = w(x, y, t) \quad (9)$$

Making use of the strain-displacement relations together with equations (7) and (9), it is found that

$$\begin{aligned}
(r_x)_1 &= (r_x)_2 = \frac{\partial \psi_x}{\partial x}, \quad (r_y)_1 = (r_y)_2 = \frac{\partial \psi_y}{\partial y} \\
(r_{xy})_1 &= (r_{xy})_2 = \frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y} \\
(r_{xz})_1 &= (r_{xz})_2 = \psi_x + \frac{\partial w}{\partial x}, \quad (r_{yz})_1 = (r_{yz})_2 = \psi_y + \frac{\partial w}{\partial y}
\end{aligned} \tag{10}$$

Hence, the moments  $M_x$ ,  $M_y$  and  $H_{xy}$  can be expressed in terms of the displacement functions  $\psi_x$ ,  $\psi_y$  and  $w$ :

$$\begin{aligned}
M_x &= D_0 \left[ \frac{\partial \psi_x}{\partial x} + \nu_0 \frac{\partial \psi_y}{\partial y} \right] \\
M_y &= D_0 \left[ \frac{\partial \psi_y}{\partial y} + \nu_0 \frac{\partial \psi_x}{\partial x} \right] \\
H_{xy} &= \frac{D_0}{2} (1 - \nu_0) \left( \frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y} \right)
\end{aligned} \tag{11}$$

The same applies to  $Q_x$  and  $Q_y$  which become

$$\begin{aligned}
Q_x &= \frac{\pi^2}{12} h \mu_0 \left( \psi_x + \frac{\partial w}{\partial x} \right) \\
Q_y &= \frac{\pi^2}{12} h \mu_0 \left( \psi_y + \frac{\partial w}{\partial y} \right)
\end{aligned} \tag{12}$$

Note that  $D_0$ ,  $\nu_0$  and  $\mu_0$  are defined as

$$D_0 = D_1 + D_2, \quad \nu_0 = \frac{D_1 \nu_1 + D_2 \nu_2}{D_0}, \quad \mu_0 = \frac{\mu_1 + \mu_2}{2} \tag{13}$$



Equations (11) and (12) are, in fact, similar to those derived in [5] for the case of a single layer homogeneous plate except that the constants  $D$ ,  $\nu$  and  $\mu$  are now replaced by  $D_0$ ,  $\nu_0$  and  $\mu_0$ .

*Governing differential equations.* Consider the elastodynamic equations of motion given by

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= \rho(z) \frac{\partial^2 u_x}{\partial t^2} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= \rho(z) \frac{\partial^2 v_y}{\partial t^2} \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= \rho(z) \frac{\partial^2 w_z}{\partial t^2}\end{aligned}\tag{14}$$

in which the mass density may vary in the thickness direction of the laminate. Multiplying the first two equations by  $z$ , expressing the stresses in terms of moments and integrating the results with respect to  $z$  from  $-h/2$  to  $h/2$  lead to

$$\begin{aligned}\frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - Q_x &= \frac{\rho_0}{12} h^3 \frac{\partial^2 \psi_x}{\partial t^2} \\ \frac{\partial H_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &= \frac{\rho_0}{12} h^3 \frac{\partial^2 \psi_y}{\partial t^2} \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q &= \bar{\rho} h \frac{\partial^2 w}{\partial t^2}\end{aligned}\tag{15}$$

in which

$$\rho_0 = \frac{1}{8} (\rho_1 + 7\rho_2), \quad \bar{\rho} = \frac{1}{2} (\rho_1 + \rho_2)\tag{16}$$

The result of inserting equations (11) and (12) into equations (15) is a system of second order partial differential equations

$$\begin{aligned} \frac{(1-\nu_0)}{2} D_0 \nabla^2 \psi_x + \frac{(1+\nu_0)}{2} D_0 \frac{\partial}{\partial x} \left( \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) - \frac{\pi^2 h}{12} \mu_0 \left( \psi_x + \frac{\partial w}{\partial x} \right) &= \frac{\rho_0}{12} h^3 \frac{\partial^2 \psi_x}{\partial t^2} \\ \frac{(1-\nu_0)}{2} D_0 \nabla^2 \psi_y + \frac{(1+\nu_0)}{2} D_0 \frac{\partial}{\partial y} \left( \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) - \frac{\pi^2 h}{12} \mu_0 \left( \psi_y + \frac{\partial w}{\partial y} \right) &= \frac{\rho_0}{12} h^3 \frac{\partial^2 \psi_y}{\partial t^2} \end{aligned} \quad (17)$$

$$\frac{\pi^2 h}{12} \mu_0 \left( \nabla^2 w + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) + q = \bar{\rho} h \frac{\partial^2 w}{\partial t^2}$$

where  $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$  is the Laplacian operator in two dimensions. Equations (17) may be combined to give a single equation

$$\begin{aligned} (D_0 \nabla^2 - \frac{\rho_0 h^3}{12} \frac{\partial^2}{\partial t^2}) (\nabla^2 - \frac{12 \bar{\rho}}{\pi^2 \mu_0} \frac{\partial^2}{\partial t^2}) w + \bar{\rho} h \frac{\partial^2 w}{\partial t^2} \\ = [1 - \frac{12 D_0}{\pi^2 h \mu_0} \nabla^2 + \frac{h^2 \rho_0}{\pi^2 \mu_0} \frac{\partial^2}{\partial t^2}] q \end{aligned} \quad (18)$$

solving for the transverse displacement  $w(x,y)$  of the laminated plate.

*Boundary conditions.* In order to derive the boundary conditions that must be specified on the crack, consider the energy stored in the laminate

$$\bar{W} = \int_{-h/2}^{h/2} W dz = \frac{1}{2} (M_x \Gamma_x + M_y \Gamma_y + H_{xy} \Gamma_{xy} + Q_x \Gamma_{xz} + Q_y \Gamma_{yz}) \quad (19)$$

in which  $\Gamma_x, \Gamma_y, \dots, \Gamma_{yz}$  are related to  $\psi_x, \psi_y$  and  $w$  as indicated in equations (10). Equations (5) and (6) may thus be applied to render

$$4\bar{W} = D_0(1+\nu_0)(r_x + r_y)^2 + \frac{\pi^2}{6} \mu_0 h(r_{yz}^2 + r_{xz}^2) + D_0(1-\nu_0)[(r_x - r_y)^2 + r_{xy}^2] \quad (20)$$

Since the physical constants  $D_0(1+\nu_0)$  and  $D_0(1-\nu_0)$  are positive,  $\bar{W}$  is a positive definite quantity. Hence,  $\bar{W}$  vanishes if and only if the equivalent strains  $r_x$ ,  $r_y$ , etc., vanish individually. Equation (20) also implies that

$$M_x = \frac{\partial \bar{W}}{\partial r_x}, \quad M_y = \frac{\partial \bar{W}}{\partial r_y}, \quad H_{xy} = \frac{\partial \bar{W}}{\partial r_{xy}} \quad (21)$$

and

$$Q_x = \frac{\partial \bar{W}}{\partial r_{xz}}, \quad Q_y = \frac{\partial \bar{W}}{\partial r_{yz}} \quad (22)$$

The kinetic energy in the laminate is

$$\bar{T} = \int_{-h/2}^{h/2} T dz = \frac{1}{2} \int_{-h/2}^{h/2} \rho(z) \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] dz$$

which, when expressed in terms of  $\psi_x$ ,  $\psi_y$  and  $w$ , takes the form

$$\bar{T} = \frac{\rho_0 h^3}{24} \left[ \left( \frac{\partial \psi_x}{\partial t} \right)^2 + \left( \frac{\partial \psi_y}{\partial t} \right)^2 \right] + \frac{\bar{\rho} h}{2} \left( \frac{\partial w}{\partial t} \right)^2 \quad (23)$$

It is now possible to write down the expression for the total energy of the laminate at time  $t$ :

$$\begin{aligned} \bar{T} + \bar{V} = & \int_{t_0}^t dt \iint \left\{ \frac{\rho_0 h^3}{24} \left[ \left( \frac{\partial \psi_x}{\partial t} \right)^2 + \left( \frac{\partial \psi_y}{\partial t} \right)^2 \right] + \frac{\bar{\rho} h}{2} \left( \frac{\partial w}{\partial t} \right)^2 \right\} dx dy \\ & + \int_{t_0}^t dt \iint \frac{\partial \bar{W}}{\partial t} dx dy + \bar{T}_0 + \bar{V}_0 \end{aligned} \quad (24)$$

where  $\bar{V}$  is the total potential energy. Note that  $\bar{T}_0$  and  $\bar{V}_0$  are the values of  $\bar{T}$  and  $\bar{V}$  corresponding to time  $t_0$ . Equation (24) may be integrated by parts and the results may be arranged to read as\*

$$\begin{aligned} \bar{T} + \bar{V} = & \int_{t_0}^t dt \int_S \left( \frac{\partial \psi_n}{\partial t} M_n + \frac{\partial \psi_s}{\partial t} H_{ns} + \frac{\partial w}{\partial t} Q_n \right) ds \\ & + \int_{t_0}^t dt \iint q \frac{\partial w}{\partial t} dx dy + \bar{T}_0 + \bar{V}_0 \end{aligned} \quad (25)$$

The above result may be interpreted as the total energy in the laminate at time  $t$  and consists of the initial energy at  $t_0$  plus the work done by the external forces along the edges and over the surfaces of the laminate during the time interval  $t-t_0$ . The initial and boundary conditions for the laminate can now be easily extracted from equation (25). They can be summarized as follows:

(1) On the laminate or crack edges: Any combination containing one member of each of the three pairs  $(\frac{\partial \psi_n}{\partial t}, M_n)$ ,  $(\frac{\partial \psi_s}{\partial t}, H_{ns})$  and  $(\frac{\partial w}{\partial t}, Q_n)$  may be specified on the crack or laminate edge.

(2) Throughout the laminate: The initial values of  $\psi_x$ ,  $\psi_y$  and  $w$  and their time derivatives need be known.

(3) Tractions and Displacements: The external load  $q$  or the displacement  $w$  on the laminate may be specified.

This completes the development of the dynamic laminate plate theory which will be used to solve a crack problem.

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\*Refer to page 45 for the derivation of equation (25).

### A CRACKED LAMINATE PLATE

As an example, consider the laminate in Figure 1 to be initially at rest and bent suddenly by a moment with a constant magnitude of  $M_0$  maintained on the crack surfaces. The conditions can be stated as

$$Q_y(x,0,t) = H_{xy}(x,0,t) = 0 \text{ for } 0 \leq |x| < \infty \quad (26)$$

and

$$M_y(x,0,t) = -M_0 H(t) \text{ for } |x| < a \text{ and } \psi_y(x,0,t) = 0 \text{ for } |x| > a \quad (27)$$

which is of the mixed type. The displacement functions are subjected to the conditions that

$$\lim_{x^2+y^2 \rightarrow \infty} [\psi_x(x,y,t), \psi_y(x,y,t), w(x,y,t)] = 0$$

No other external forces or constraints are present.

*Laplace transform.* The governing equations (17) will be solved by introducing the Laplace transform pair

$$f^*(p) = \int_0^{\infty} f(t) e^{-pt} dt \quad (28)$$

$$f(t) = \frac{1}{2\pi i} \int_{Br} f^*(p) e^{pt} dp$$

where the second integral is over the Bromwich path. Applying the first of equations (28) to (17) yields

$$\begin{aligned}
\frac{(1-\nu_0)}{2} D_0 \nabla^2 \psi_x^* + \frac{(1+\nu_0)}{2} D_0 \frac{\partial}{\partial x} \left( \frac{\partial \psi_x^*}{\partial x} + \frac{\partial \psi_y^*}{\partial y} \right) - \frac{\pi^2 h}{12} \mu_0 \left( \psi_x^* + \frac{\partial w^*}{\partial x} \right) &= \frac{\rho_0 h^3}{12} p^2 \psi_x^* \\
\frac{(1-\nu_0)}{2} D_0 \nabla^2 \psi_y^* + \frac{(1+\nu_0)}{2} D_0 \frac{\partial}{\partial y} \left( \frac{\partial \psi_x^*}{\partial x} + \frac{\partial \psi_y^*}{\partial y} \right) - \frac{\pi^2 h}{12} \mu_0 \left( \psi_y^* + \frac{\partial w^*}{\partial y} \right) &= \frac{\rho_0 h^3}{12} p^2 \psi_y^* \\
\frac{\pi^2 h}{12} \mu_0 \left( \nabla^2 w^* + \frac{\partial \psi_x^*}{\partial x} + \frac{\partial \psi_y^*}{\partial y} \right) &= \bar{\rho} h p^2 w^*
\end{aligned} \tag{29}$$

The analysis may be simplified by letting

$$\psi_x^* = \frac{\partial \phi^*}{\partial x} + \frac{\partial H^*}{\partial y}, \quad \psi_y^* = \frac{\partial \phi^*}{\partial y} - \frac{\partial H^*}{\partial x} \tag{30}$$

such that equations (29) simplify to

$$\begin{aligned}
\frac{\partial}{\partial x} \{ \nabla^2 \phi^* - (R\delta_0^4 + S^{-1})\phi^* - S^{-1}w^* \} + \frac{1-\nu_0}{2} \frac{\partial}{\partial y} (\nabla^2 - \omega^2) H^* &= 0 \\
\frac{\partial}{\partial y} \{ \nabla^2 \phi^* - (R\delta_0^4 + S^{-1})\phi^* - S^{-1}w^* \} - \frac{1-\nu_0}{2} \frac{\partial}{\partial x} (\nabla^2 - \omega^2) H^* &= 0 \\
\nabla^2 (\phi^* + w^*) - S\delta_0^4 w^* &= 0
\end{aligned} \tag{31}$$

The new quantities introduced in equations (31) are defined as

$$R = \frac{h^2}{12}, \quad S = \frac{12D_0}{\pi^2 h \mu_0}, \quad \delta_0^4 = \frac{\rho_0 h p^2}{D_0}, \quad \bar{\delta}_0^4 = \frac{\bar{\rho} h p^2}{D_0} \tag{32}$$

and

$$\omega^2 = \frac{2(R\delta_0^4 + S^{-1})(D_1 + D_2)}{(1-\nu_1)D_1 + (1-\nu_2)D_2} \tag{33}$$

Furthermore, if

$$\phi^* = (\beta - 1)w^* \quad (34)$$

is introduced into equations (31), it can be shown that

$$\nabla^2 w^* - \alpha^2 w^* = 0 \quad (35)$$

while  $\alpha$  and  $\beta$  are given by

$$\alpha^2 = R\delta_0^4 + S^{-1} + \frac{S^{-1}}{\beta - 1}, \quad \beta = \frac{S\delta_0^4}{\alpha^2} \quad (36)$$

Consequently, the functions  $\psi_x^*$  and  $\psi_y^*$  in equations (30) become

$$\psi_x^* = (\beta_1 - 1) \frac{\partial w_1^*}{\partial x} + (\beta_2 - 1) \frac{\partial w_2^*}{\partial x} + \frac{\partial H^*}{\partial y} \quad (37)$$

$$\psi_y^* = (\beta_1 - 1) \frac{\partial w_1^*}{\partial y} + (\beta_2 - 1) \frac{\partial w_2^*}{\partial y} - \frac{\partial H^*}{\partial x}$$

and  $w^*$  may be written as

$$w^* = w_1^* + w_2^* \quad (38)$$

In equations (37),  $\beta_1$  and  $\beta_2$  are given as

$$\beta_{1,2} = (R\delta_0^4 + S^{-1})^{-1} \alpha_{2,1}^2 \quad (39)$$

in which

$$\alpha_{1,2}^2 = \frac{1}{2} \{ (R\delta_0^4 + S\delta_0^4) \pm [(R\delta_0^4 - S\delta_0^4)^2 - 4\delta_0^4]^{1/2} \} \quad (40)$$

It is now apparent that once  $H^*$ ,  $w_1^*$  and  $w_2^*$  are found from

$$(\nabla^2 - \omega^2)H^* = 0, (\nabla^2 - \alpha_1^2)w_1^* = 0, (\nabla^2 - \alpha_2^2)w_2^* = 0 \quad (41)$$

the problem is basically solved in the Laplace transform plane.

*Integral equation.* Taking advantage of the symmetry condition with respect to the y-axis, it is not difficult to show that the following integrals

$$\begin{aligned} w_1^*(x, y, p) &= \frac{2}{\pi} \int_0^\infty B^{(1)}(s, p) \cos(sx) e^{-\gamma_1 y} ds \\ w_2^*(x, y, p) &= \frac{2}{\pi} \int_0^\infty B^{(2)}(s, p) \cos(sx) e^{-\gamma_2 y} ds \\ H^*(x, y, p) &= \frac{2}{\pi} \int_0^\infty B^{(3)}(s, p) \sin(sx) e^{-\gamma_3 y} ds \end{aligned} \quad (42)$$

satisfy equations (41) provided that

$$\gamma_{1,2} = (s^2 + \alpha_{1,2}^2)^{1/2}, \gamma_3 = (s^2 + \omega^2)^{1/2} \quad (43)$$

The unknowns  $B^{(1)}$ ,  $B^{(2)}$  and  $B^{(3)}$  must be determined from the boundary conditions in equations (26) and (27) whose Laplace transform are

$$Q_y^*(x, 0, p) = H_{xy}^*(x, 0, p) = 0 \text{ for } 0 < x < \infty \quad (44)$$

and

$$M_y^*(x, 0, p) = -\frac{M_0}{p} \text{ for } 0 < x < a \text{ and } \psi_y^*(x, 0, p) = 0 \text{ for } x > a \quad (45)$$

The appropriate quantities in equations (44) and (45) may be obtained by first putting equations (42) into (37) and (38). This gives



$$\begin{aligned}
\psi_x^* &= -\frac{2}{\pi} \int_0^{\infty} \{s[(\beta_1-1)B^{(1)}(s,p)e^{-\gamma_1 y} + (\beta_2-1)B^{(2)}(s,p)e^{-\gamma_2 y}] \\
&\quad + \gamma_3 B^{(3)}(s,p)e^{-\gamma_3 y}\} \sin(sx) ds \\
\psi_y^* &= -\frac{2}{\pi} \int_0^{\infty} \{(\beta_1-1)\gamma_1 B^{(1)}(s,p)e^{-\gamma_1 y} + (\beta_2-1)\gamma_2 B^{(2)}(s,p)e^{-\gamma_2 y} \\
&\quad + sB^{(3)}(s,p)e^{-\gamma_3 y}\} \cos(sx) ds
\end{aligned} \tag{46}$$

and

$$w^* = \frac{2}{\pi} \int_0^{\infty} \{B^{(1)}(s,p)e^{-\gamma_1 y} + B^{(2)}(s,p)e^{-\gamma_2 y}\} \cos(sx) ds \tag{47}$$

The Laplace transform of equations (11) and (12) will clearly involve  $\psi_x^*$ ,  $\psi_y^*$  and  $w^*$ . Equations (46) and (47) and equations (45) can be satisfied if the function  $C(s,p)$  obeys the dual integral equations

$$\begin{aligned}
\int_0^{\infty} C(s,p) \cos(sx) ds &= 0 \quad x \geq a \\
\int_0^{\infty} sG(s,p)C(s,p) \cos(sx) ds &= \frac{\pi M_0}{D_0(1-v_0^2)p} \quad x < a
\end{aligned} \tag{48}$$

with  $G(s,p)$  being a known function

$$\begin{aligned}
\frac{(1-v_0^2)}{2} G(s,p) &= \{(1-\beta_1)(\gamma_1^2 - v_0^2 s^2)^2 / (s\gamma_1) - (1-\beta_2)(\gamma_2^2 - v_0^2 s^2)^2 / (s\gamma_2) \\
&\quad - 2s\gamma_3(1-v_0)(\alpha_1^2 - \alpha_2^2)/\omega^2\} / (\alpha_1^2 - \alpha_2^2)
\end{aligned} \tag{49}$$

The conditions in equations (44) may be used to relate the functions  $B^{(1)}$ ,  $B^{(2)}$  and  $B^{(3)}$  to  $C(s,p)$ :

$$\begin{aligned} B^{(1)}(s,p) &= \frac{(1-v_0)s^2 + \alpha_1^2}{\gamma_1(\alpha_1^2 - \alpha_2^2)} C(s,p) \\ B^{(2)}(s,p) &= - \frac{(1-v_0)s^2 + \alpha_2^2}{\gamma_2(\alpha_1^2 - \alpha_2^2)} C(s,p) \\ B^{(3)}(s,p) &= \frac{s(1-v_0)(\beta_2 - \beta_1)}{\alpha_1^2 - \alpha_2^2} C(s,p) \end{aligned} \quad (50)$$

Without going into details, the solution for equations (48) is of the form [6]

$$C(s,p) = \frac{\pi M_0 a^2}{D_0(1-v_0^2)p} \int_0^1 \sqrt{\xi} \psi(\xi,p) J_0(sa\xi) d\xi \quad (51)$$

where  $J_0$  is zero order Bessel function of the first kind and the function  $\psi^*(\xi,p)$  can be found from a Fredholm integral equation of the second kind:

$$\psi^*(\xi,p) + \int_0^1 L(\xi,\eta,p) \psi^*(\eta p) d\eta = \sqrt{\xi} \quad (52)$$

The kernel  $L(\xi,\eta,p)$  is symmetric in  $\xi$  and  $\eta$  and takes the form

$$L(\xi,\eta,p) = \sqrt{\xi\eta} \int_0^\infty s \left[ G\left(\frac{s}{a}, p\right) - 1 \right] J_0(s\xi) J_0(s\eta) ds \quad (53)$$

Equation (52) can be evaluated numerically for  $\psi^*(\xi,p)$  in the Laplace transform domain and then inverted into the time domain by using the second of equations (28).

*Dynamic moment intensity factor.* The time dependence of the solution may be recovered by two different procedures. The first is to apply the Laplace inver-

sion formula to the quantities of interest and obtain the complete solution as a function of time. Such an approach is not only cumbersome and can often result in a considerable amount of difficulties in numerical calculations. In fracture mechanics, since it is only necessary to focus attention on the state of affairs near the crack front, Sih et al [7] have suggested to obtain the asymptotic stress solution in the Laplace transform domain such that the time inversion is applied only to the first term of the stress expansion near the crack tip. This approach has greatly simplified the analysis and will be used here.

The local solution may be found by expanding the integral in equation (51) for  $C(s,p)$  for large values of the argument  $s$ . Once the moments  $M_x^*$ ,  $M_y^*$  and  $H_{xy}^*$  are expressed in terms of  $C(s,p)$ , the resulting integrals may be evaluated to give the asymptotic expansions:

$$\begin{aligned} M_x^*(r, \theta, p) &= \frac{K_1^*(p)}{\sqrt{2r}} \cos \frac{\theta}{2} \{1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\} + O(r^0) \\ M_y^*(r, \theta, p) &= \frac{K_1^*(p)}{\sqrt{2r}} \cos \frac{\theta}{2} \{1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\} + O(r^0) \\ H_{xy}^*(r, \theta, p) &= \frac{K_1^*(p)}{\sqrt{2r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^0) \end{aligned} \quad (54)$$

where  $Q_x^*$  and  $Q_y^*$  are nonsingular and remain finite as  $r \rightarrow 0$ , i.e.,

$$Q_x^* = Q_y^* = O(r^0) \quad (55)$$

The polar coordinates  $r$  and  $\theta$  are measured from the crack front as shown in Figure 1. The parameter

$$K_1^*(p) = M_0 \sqrt{a} \frac{\psi^*(1,p)}{p} \quad (56)$$

is the Laplace transform of the dynamic moment intensity factor and  $\psi^*(1,p)$  denotes the values of the function  $\psi^*(\xi,p)$  near the crack border  $\xi=1$ .

Applying the Laplace inversion theorem to equations (54) yields the solution as a function of time:

$$\begin{aligned} M_x(r,\theta,t) &= \frac{K_1(t)}{\sqrt{2r}} \cos \frac{\theta}{2} \{1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\} + O(r^0) \\ M_y(r,\theta,t) &= \frac{K_1(t)}{\sqrt{2r}} \cos \frac{\theta}{2} \{1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\} + O(r^0) \\ H_{xy}(r,\theta,t) &= \frac{K_1(t)}{\sqrt{2r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^0) \end{aligned} \quad (57)$$

The dynamic moment intensity factor  $K_1(t)$  may be computed from

$$K_1(t) = \frac{M_0 \sqrt{a}}{2\pi i} \int_{Br} \frac{\psi^*(1,p)}{p} e^{pt} dp \quad (58)$$

once  $\psi^*(1,p)$  is known.

*Numerical results.* Since the procedure for solving the Fredholm integral equation is already well known, it is not necessary to cover the details. The numerical values of  $\psi^*(1,p)$  in equation (58) are given in Figures 2 to 4 for the three different values of  $\mu_2/\mu_1 = 0.1, 1.0$  and  $10.0$ . The Poisson's ratio and mass density for the layers are assumed to be the same as their variations in the thickness direction do affect the results appreciably. The function  $\psi^*(1,p)$  is seen to increase monotonically with  $c_{21}/p a$  where  $c_{21} = (\mu_1/\rho_1)^{1/2}$  is the shear wave speed of the material in the outer layers.

As an indication of the load intensity transmitted to the crack edge region as a function of time, the moment intensity factor  $K_I(t)$  will be computed from equation (58) by using the results in Figures 2 to 4. Figures 5 to 7 display the variations of the normalized quantity  $K_I(t)/M_0\sqrt{a}$  with the dimensionless time parameter  $c_{21}t/a$  for  $\mu_2/\mu_1 = 0.1, 1.0$  and  $10.0$  while the crack length to laminate thickness ratio  $2a/h$  takes on the values of 1, 2 and 4. Generally speaking,  $K_I(t)$  tends to increase with time reaching a peak and then acquires an oscillatory character. The peak value of  $K_I(t)$  appears to be inversely proportional to the ratio of  $2a/h$ , i.e.,  $K_I(t)$  maximum at  $2a/h = 1$  is larger than that at  $2a/h = 4$ . The moment intensity tends to decrease as the crack length is increased. Also,  $K_I(t)$  maximum occurs earlier when the shear moduli in the outer layers of the laminate is larger than those in the inner layers. Refer to the curves in Figure 7 for  $\mu_2/\mu_1 > 1$  and those in Figure 5 for  $\mu_2/\mu_1 < 1$ . The influence of  $\mu_2/\mu_1$  can be best illustrated by fixing the ratio of  $2a/h$  and use  $\mu_2/\mu_1$  as a varying parameter. Figure 8 shows a plot of  $K_I(t)/M_0\sqrt{a}$  versus  $c_{21}t/a$  as  $\mu_2/\mu_1$  takes the values 0.1, 1.0 and 10.0. It is clear that the crack edge moment intensity can be reduced by letting  $\mu_2 < \mu_1$ , i.e., making the shear moduli of the inner layers to be larger than the moduli of the outer layers.

#### CONCLUDING REMARKS

A dynamic laminated plate theory is developed with emphases placed on obtaining effective solution for the crack configuration where the  $1/\sqrt{r}$  stress singularity and the condition of plane strain are preserved. The radial distance  $r$  is measured from the crack edge. Although each layer in the laminate is assumed to be isotropic, it is a simple extension to include anisotropy simulating the directional properties of fiber reinforcement. This additional com-

plexity was not thought to be necessary in this preliminary analysis.

Several revealing conclusions can be made from the numerical results of the example on the sudden bending of a cracked laminate when compared with a single layer homogeneous plate.

(1) The crack moment intensity tends to decrease as the crack length to laminate plate thickness is increased. Hence, a laminated plate has the desirable feature of stabilizing a through crack as it increases its length at constant load.

(2) The level of the average load intensity transmitted to a through crack can be reduced by making the inner layers to be stiffer than the outer layers.

The foregoing comments are strictly based on the concept of moment intensity factor as used in the theory of fracture mechanics. In the normal course of design, other considerations must also be accounted for. However, the point has been made that the present theory, although approximate, is useful for analyzing laminate failure due to crack propagation.

#### ACKNOWLEDGEMENTS

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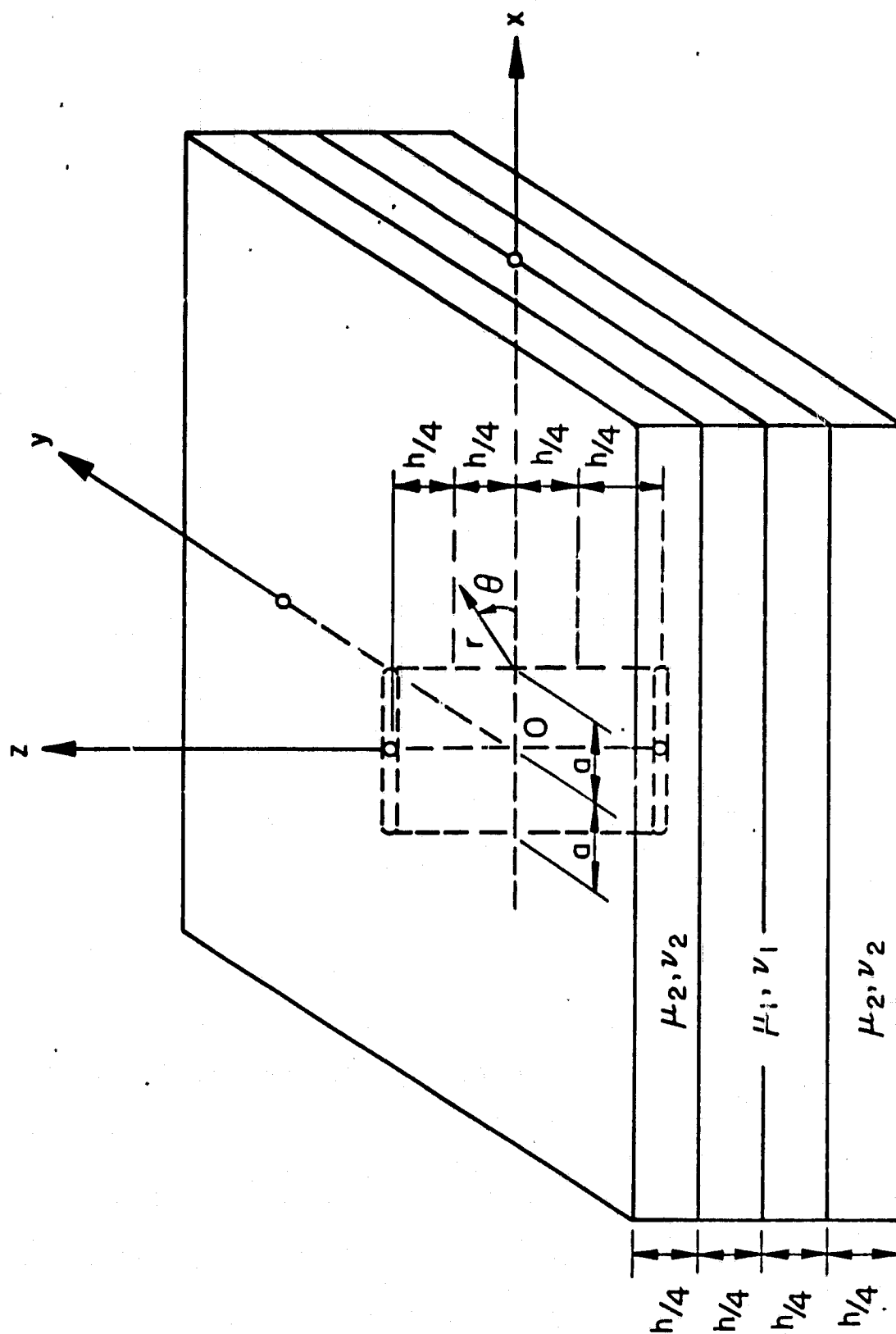


Figure 1 - A symmetrically layered plate with a through crack



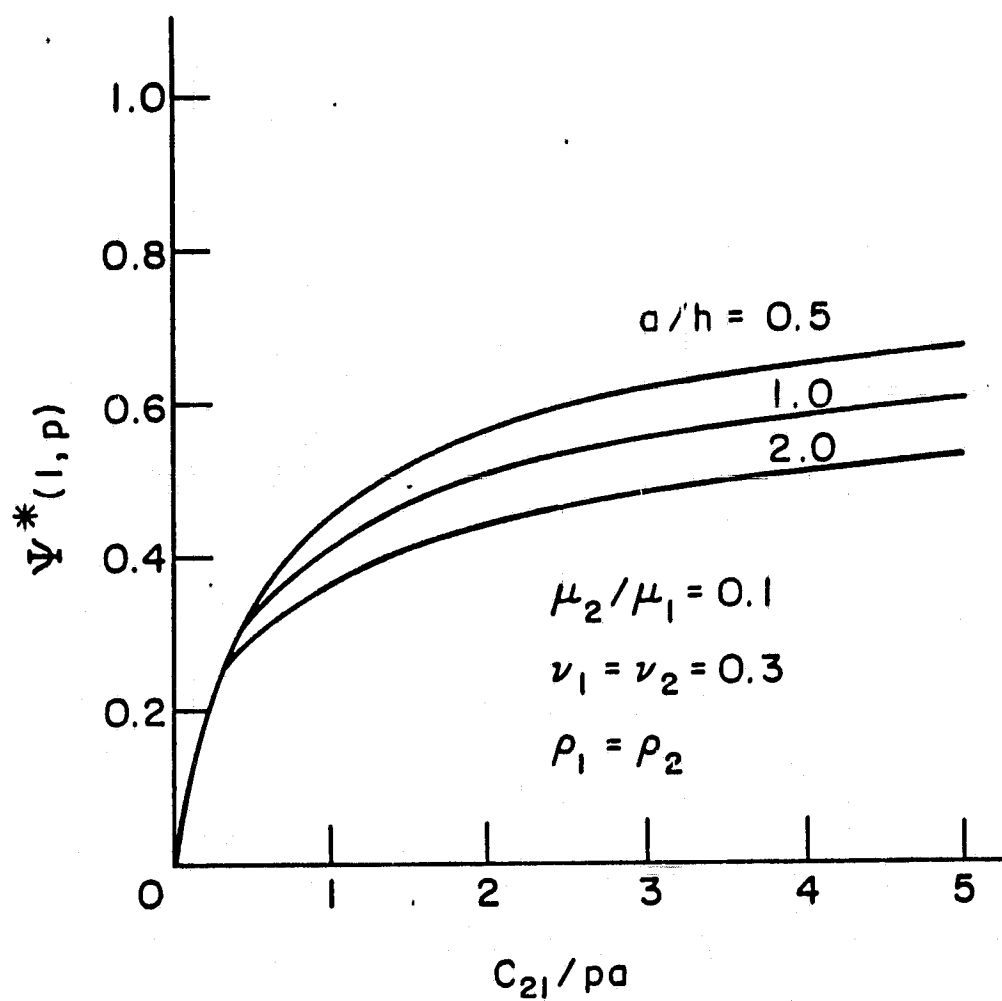


Figure 2 - Numerical values of  $\Psi^*(1,p)$  as a function of  $c_{21}/pa$  for  $\mu_2/\mu_1 = 0.1$

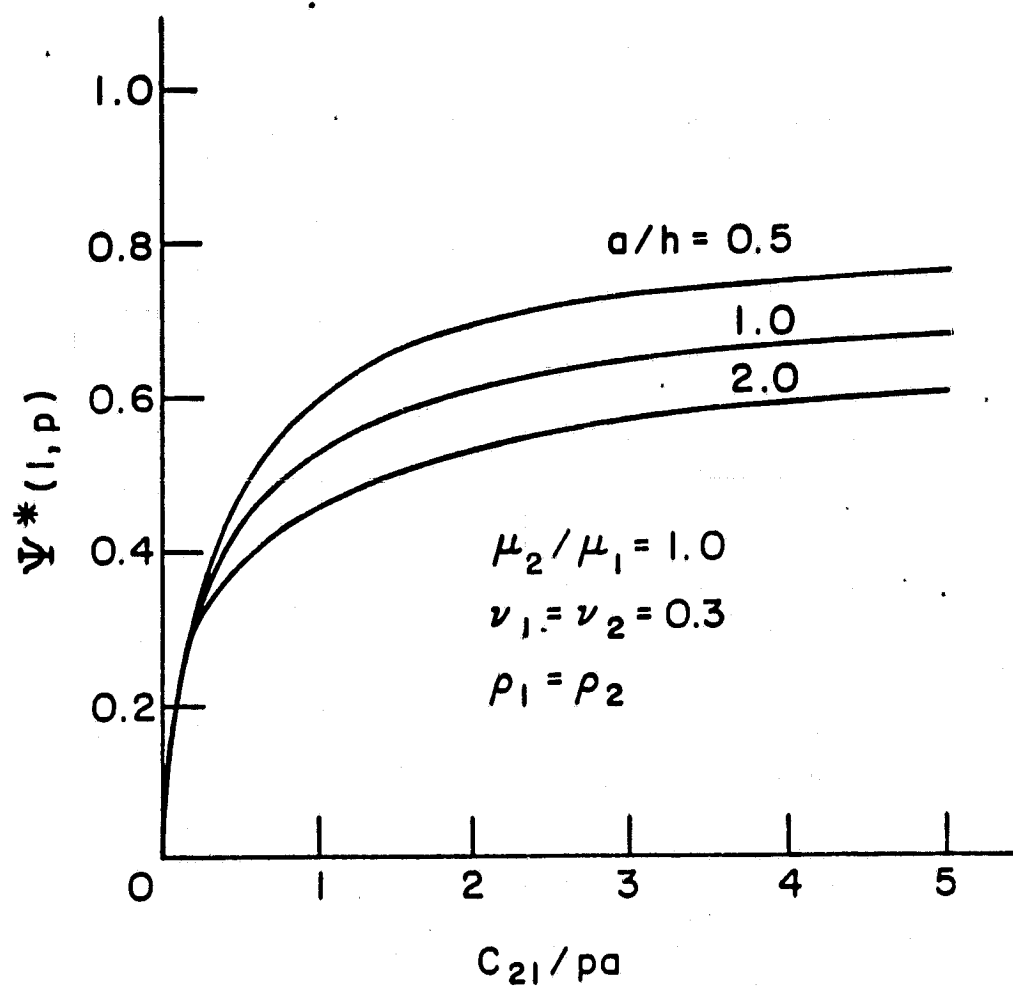


Figure 3 - Numerical values of  $\Psi^*(1,p)$  as a function of  $c_{21}/pa$  for  $\mu_2/\mu_1 = 1.0$

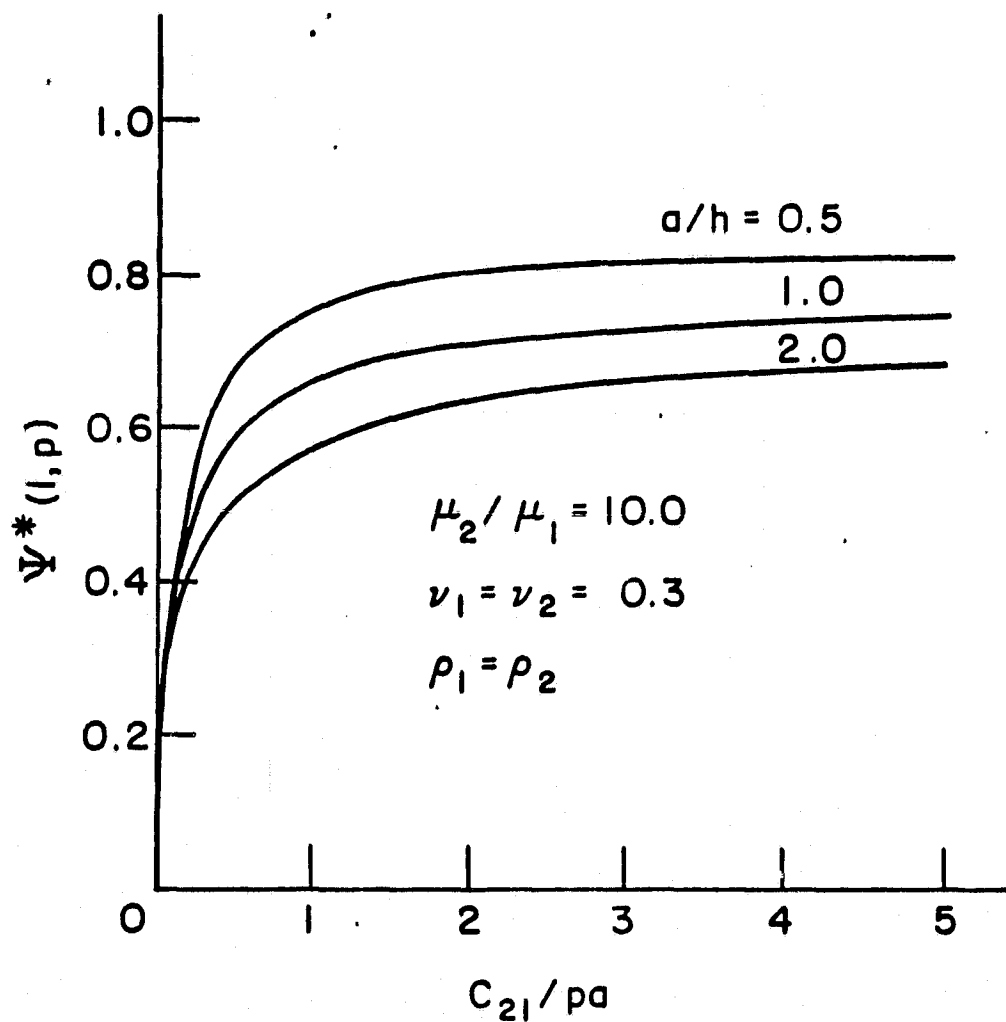


Figure 4 - Numerical values of  $\Psi^*(1,p)$  as a function of  $c_{21}/pa$  for  $\mu_2/\mu_1 = 10.0$

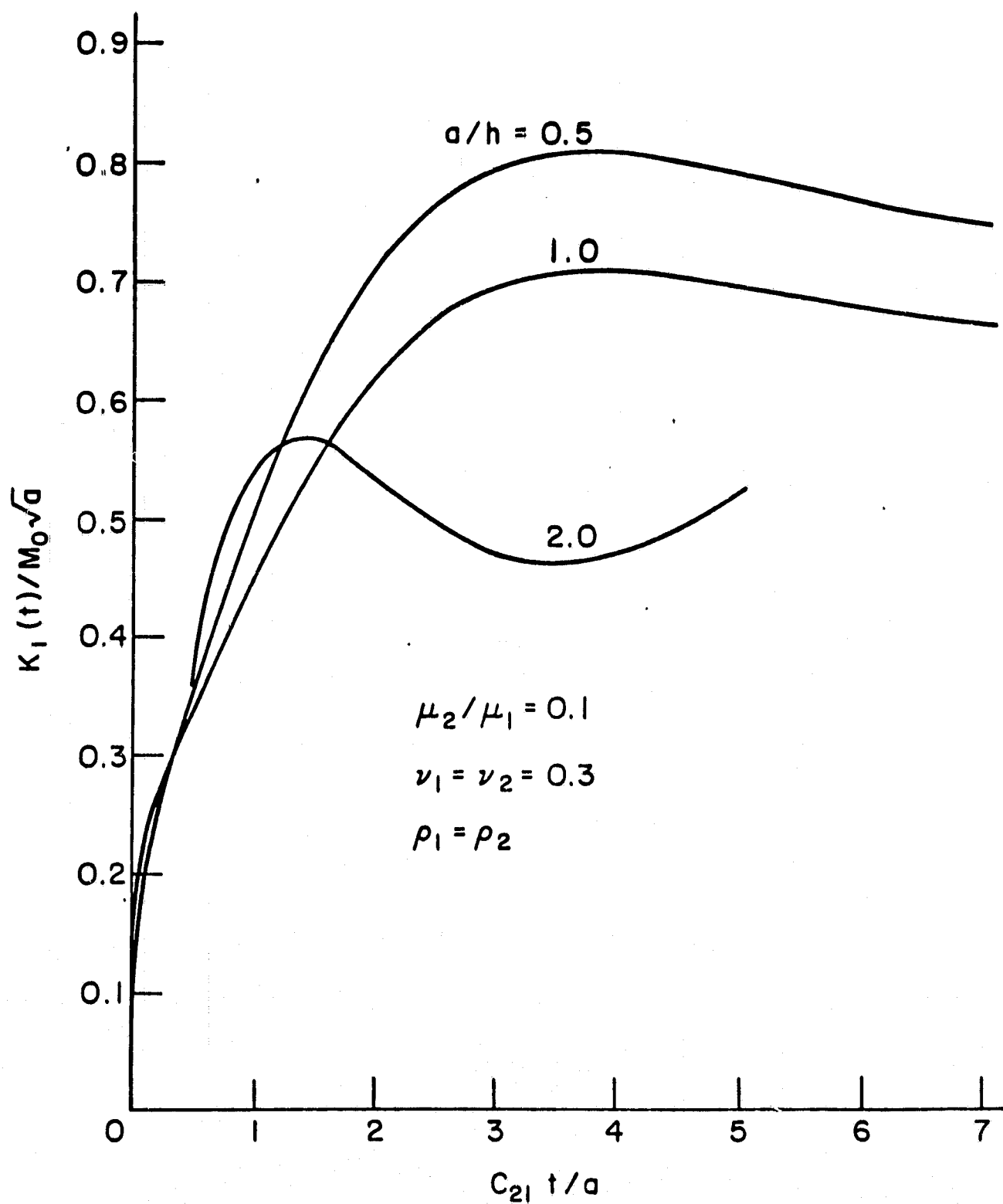


Figure 5 - Normalized moment intensity factor as a function of  $C_{21}t/a$  for  $\mu_2/\mu_1 = 0.1$

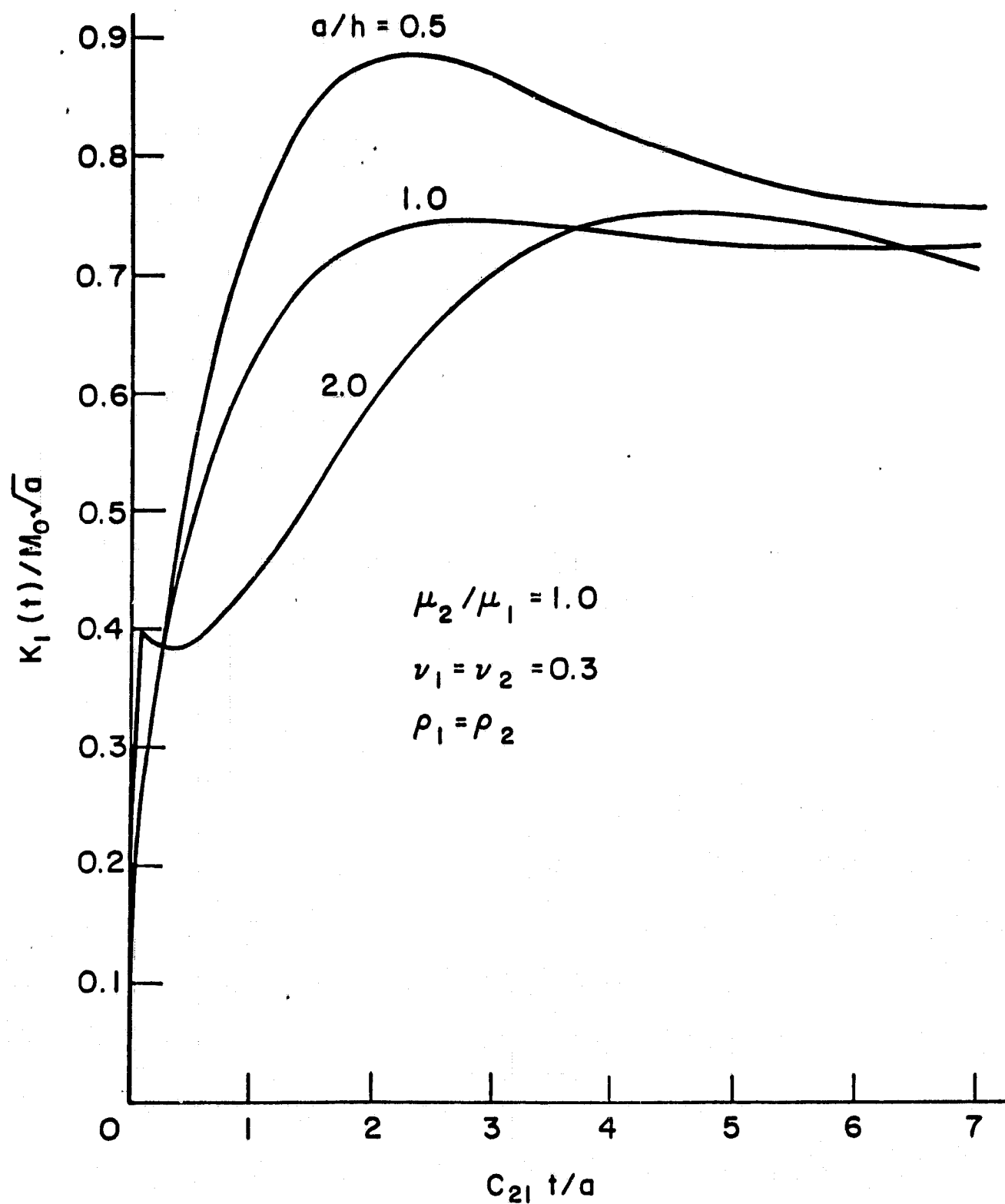


Figure 6 - Normalized moment intensity factor as a function of  $C_{21}t/a$  for  $\mu_2/\mu_1 = 1.0$

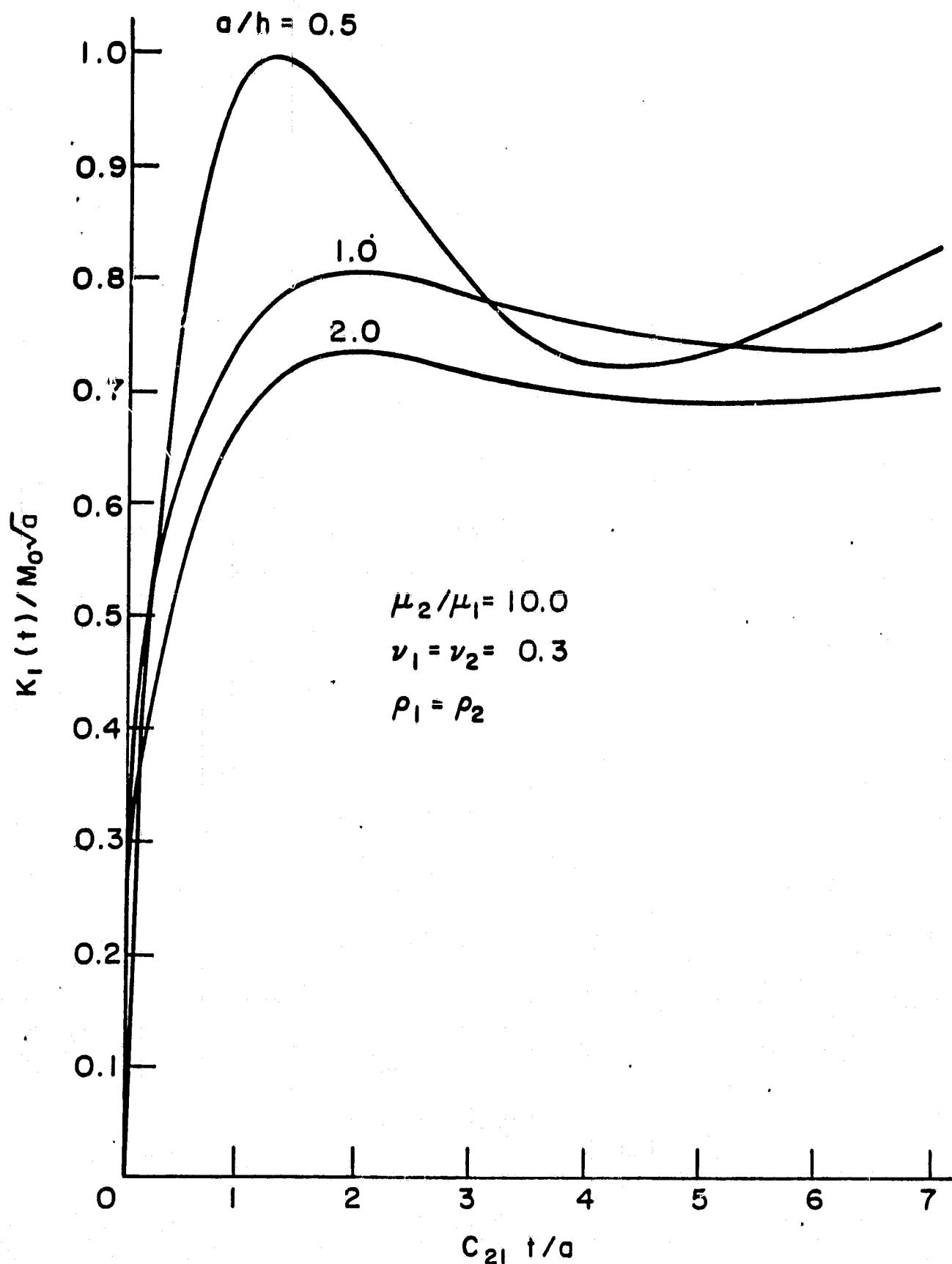


Figure 7 - Normalized moment intensity factor as a function of  $C_{21}t/a$  for  $\mu_2/\mu_1 = 10.0$

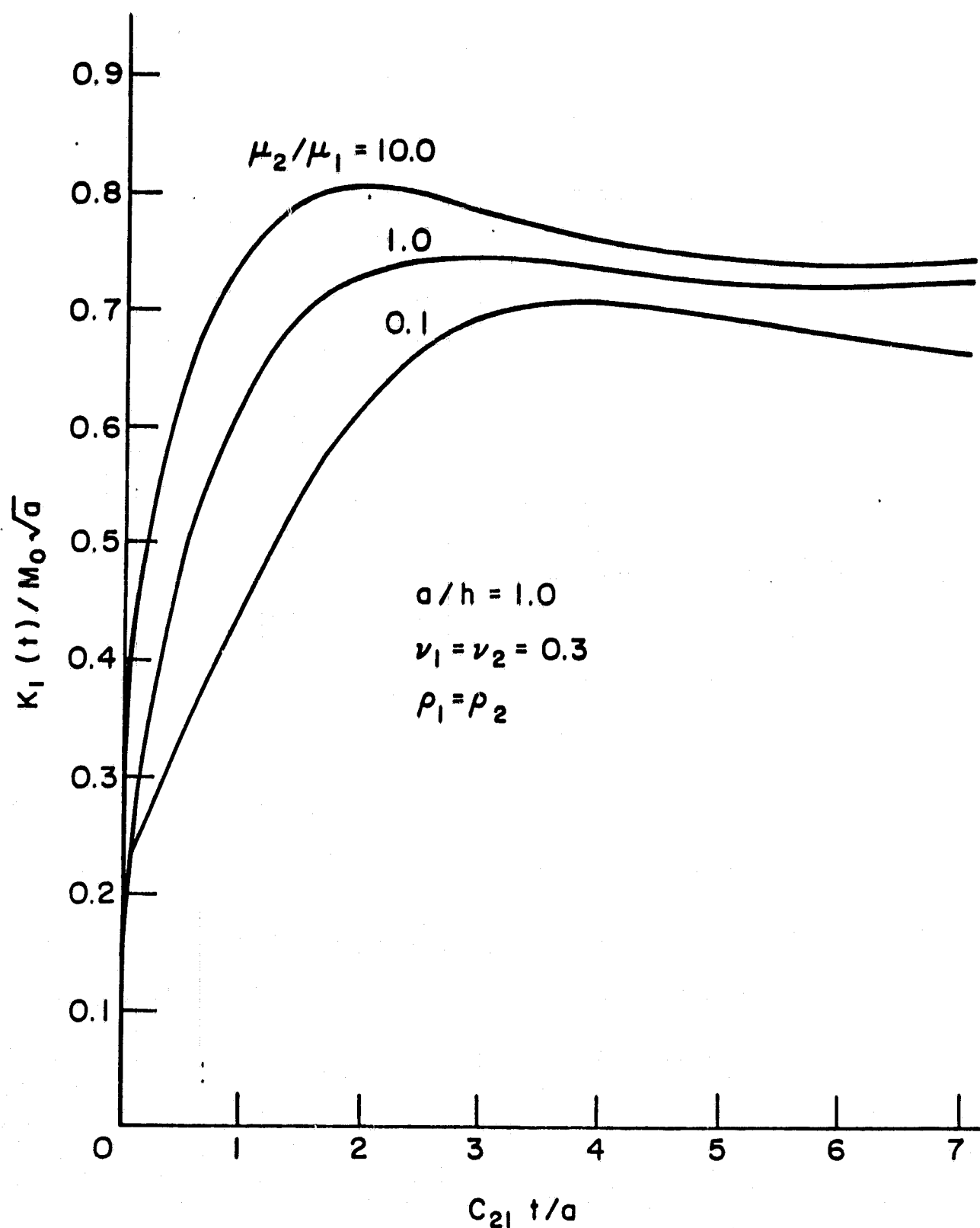


Figure 8 - Normalized moment intensity factor as a function of  $C_{21} t/a$  for  $a/h = 1.0$

# Computer Program for Bending of Cracked Laminate Plates

```

PROGRAM HETA(INPUT,OUTPUT)
REAL NON(4),F(4,4,1),G(4,4),D(4),PT(4)
REAL B(4),C(4)
REAL LP(19),DTA(19)
EQUIVALENCE (NON,B)
COMMON K1,K2,K3,K4
COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
LP(1)=0.0
DTA(1)=0.0
READ 2,K1,K2,K3,K4
2 FORMAT(12)
* K1 = ORDER OF SYSTEM OF EQUATIONS
* K2 = NO. OF DISTINCT KERNELS
* K3 = NO. OF DATA POINTS
* K4 = NO. OF DATA SETS TO BE EVALUATED
* SET UP DATA POINTS
  AK=K3
  DO 5 N=1,K3
  AN=N
  5 PT(N)=AN/AK
* SET UP INTEGRATION MATRIX
  M=K3-2
  N=K3-1
  A=K3
  A=1./(3.*A)
  DO 10 K=2,M,2
  10 D(K)=2.*A
  DO 15 K=1,N,2
  15 D(K)=4.*A
  D(K3)=A
* CALCULATE NONHOMOGENEOUS TERMS
  RHS=1.0
  DO 22 I=1,K2
  PRINT 9
  9 FORMAT(1H1)
  DO 999 II=1,K4
  DO 35 N=1,K3
  35 NON(N)=RHS*SQRT(PT(N))
* CALCULATE KERNEL MATRICES
  CALL CONST(I)
  DO 20 N=1,K3
  DO 20 M=1,K3
  F(M,N,I)=FII(I,PT(M),PT(N))
  20 CONTINUE
  CALL CHANGE(F,G,D,I)
  CALL LINEQ(G,B,C,K3)
  DO 40 L=1,K3
  PRINT 6,PT(L),NON(L)
  6 FORMAT(5X,F8.4,F15.6)
  40 CONTINUE
  LP(II+1)=NON(K3)
  DTA(II+1)=P
  999 CONTINUE
  CALL LAPINV(DTA,LP)
  22 CONTINUE
  END

```



```

FUNCTION SIMP(I,A,H)
COMMON/AUX/H,P,PK1,PK2,HMU,X,Y
MXYZ=2**15
DFL=0.25*(H-A)
IF(DEL)40,45,50
45 SIMP=0.0
RETURN
50 CONTINUE
SA=Z(I,A)+7(I,H)
SH=Z(I,A+2.*DFL)
SC=Z(I,A+DFL)+7(I,A+3.*DFL)
S1=(DEL/3.)*(SA+2.*SH+4.*SC)
IF(S1.EQ.0.0) GO TO 45
K=P
35 SH=SH+SC
DEL=0.5*DEL
SC=Z(I,A+DEL)
J=K-1
DO 5 N=3,J,2
AN=N
5 SC=SC+Z(I,A+AN*DEL)
S2=(DEL/3.)*(SA+2.*SH+4.*SC)
DIF=ABS((S2-S1)/S1)
ER=0.01
IF(DIF-ER)30,25,25
30 SIMP=S2
RETURN
25 K=2*K
S1=S2
IF(K-MXYZ)35,35,40
40 PRINT 42,I,A,H
42 FORMAT(5X,' INT. DOES NOT CONVERGE ',I3,2F9.4)
PRINT 60,X,Y
60 FORMAT(2F10.5)
DO 70 J=1,10
DIP=J
DIP=DIP/10.
W=Z(I,DIP)
PRINT 60,W
70 CONTINUE
CALL EXIT
END

```

```

SUBROUTINE LINEQ(A,B,T,N)
REAL A(N,N),B(N),T(N)
DO 5 I=2,N
5  A(I,1)=A(I,1)/A(1,1)
DO 10 K=2,N
M=K-1
DO 15 I=1,M
15  T(I)=A(I,K)
DO 20 J=1,M
A(J,K)=T(J)
J1=J+1
DO 20 I=J1,N
T(I)=T(I)-A(I,J)*A(J,K)
20  CONTINUE
A(K,K)=T(K)
IF(K.EQ.N) GO TO 10
M=K+1
DO 25 I=M,N
25  A(I,K)=T(I)/A(K,K)
10  CONTINUE
* BACK SUBSTITUTE
DO 31 I=1,N
T(I)=B(I)
M=I+1
IF(M.GT.N) GO TO 31
DO 30 J=M,N
R(J)=B(J)-A(J,I)*T(I)
30  CONTINUE
31  CONTINUE
DO 35 I=1,N
K=N+1-I
R(K)=T(K)/A(K,K)
K1=K-1
IF(K1.EQ.0) GO TO 35
DO 36 J=1,K1
J=K-J1
T(J)=T(J)-A(J,K)*R(K)
36  CONTINUE
35  CONTINUE
RETURN
END

```

```

FUNCTION FU(I,A,R)
COMMON/ALIX/H,P,PK1,PK2,RMU,X,Y
X=A
Y=R
IF(A*R)5,10,5
10 FU=0.0
RETURN
5 SUM=SIMP(I,0.0,5.0)
FR=0.01
DEL =5.0
20 UP=DEL+5.0
ADDL=SIMP(I,DEL,UP)
DEL =UP
TEST=ABS(ADDL/SUM)
SUM=SUM+ADDL
IF(1TEST-FR)15,20,20
15 FU=SQRT(X*Y)*SUM
RETURN
END

```

```

SUBROUTINE CHANGE(F,G,D,I)
REAL F(4,4,1),G(4,4),D(4)
COMMON K1,K2,K3,K4
DO 10 N=1,K3
DO 10 M=1,K3
G(M,N) =F(M,N,I)*D(N)
10 CONTINUE
DO 20 N=1,K3
G(N,N)=G(N,N)+1.0
20 RETURN
END

```

```

FUNCTION BESJO(A)
  IF (A-3.)5.5.10
5  H=A*A/9.
  W=1.-2.2499997*B
  Z=R*B
  W=W+1.2656208*Z
  Z=Z*B
  W=W-.3163866*Z
  Z=Z*B
  W=W+.0444479*Z
  Z=Z*B
  W=W-.0039444*Z
  Z=Z*B
  BESJO=W+.00021*Z
  RETURN
10 H=3./A
  W=.79788456-.00000077*B
  V=A-.78539816-.04166397*B
  Z=R*B
  W=W-.0055274*Z
  V=V-.00003954*Z
  Z=Z*B
  W=W-.00009512*Z
  V=V+.00262573*Z
  Z=Z*B
  W=W+.00137237*Z
  V=V-.00054125*Z
  Z=Z*B
  W=W-.00072805*Z
  V=V-.00029333*Z
  Z=Z*B
  W=W+.00014476*Z
  V=V+.00013558*Z
  BESJO=W/SQRT(A)*COS(V)
  RETURN
END

```

```

1      SUBROUTINE CONST(I)
      COMMON/AUX/H,P,PK1,PK2,MMU,X,Y
      PK1=0.3
      PK2=0.3
      MMU=50.0
      H=1.0
      READ 2,P
      ? FORMAT(F10.5)
      HH=1./H
      PRINT 1,MMU,PK1,PK2,HH,P
      1 FORMAT(//////5X,* MU?/MU1 =*F6.2,* NU1 =*F4.2,* NU2 =*F4.2//5X,*
      1A/H =*F4.2,* C?1/P? =*F4.2//)
      RETURN
      END

```

# SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS  
3 CONST

VARIABLES	SN	TYPE	RELOCATION	0	H	REAL	INTEGER	*UNUSED
4 MMU		REAL	AUX	0	0			
55 HH		REAL		0	0			
1 P		REAL	AUX	2	PK1			
3 PK2		REAL	AUX	5	X			
4 Y		REAL	AUX					

FILE NAMES	MODE	INPUT	OUTPUT	FMT
	FMT			

STATEMENT LABELS	27	1	FMT	25	2	FMT

COMMON BLOCKS	LENGTH	AUX
	7	

STATISTICS

PROGRAM LENGTH	56H	46
SCN LABELED COMMON LENGTH	7B	7
47600H SCN USED		

```

FUNCTION Z(I,S)
COMMON/AUX/H,P,PK1,PK2,RMU,X,Y
COMPLEX AC,AL1,AL2,SA,SH
COMPLEX GA,GB,RA,RH,RC,F,G
PI=3.1415926
PP=P*P
P=H*H/12.
AA=1.+7.*RMU*(1.-PK1)/(1.-PK2)
DEO=48.*(1.-PK1)/(PP*H*H*AA)
SS=H*H*AA/(2.*PI*PI*(1.-PK1)*(1.+BMI))
XNUO=(PK1-PK2+AA*PK2)/AA
AD=(R+SS)*DEO
ZZ=(R-SS)*(R-SS)*DEO*DEO-4.*DEO
G=CMPLX(ZZ,0.0)
AC=CSQRT(G)
AL1=0.5*(AH+AC)
AL2=0.5*(AH-AC)
AL3=2.*(P*DEO+1./SS)/(1.-XNUO)
SA=AL2/(R*DEO+1./SS)
SB=AL1/(R*DEO+1./SS)
GA=CSQRT(S*S+AL1)
GB=CSQRT(S*S+AL2)
GC=SQRT(S*S+AL3)
RA=2./(1.-XNUO*XNUO)/(AL1-AL2)
RH=GA*GA-XNUO*S*S
RC=GB*GB-XNUO*S*S
F=RA*((1.-SA)*RH*RH/GA-(1.-SB)*RC*RC/GB-2.*S*S*GC*(1.-XNUO)*(AL1-
1/2)/AL3)
Q=PFAL(F)
QA=AIMAG(F)
IF(QA-0.0)5,10,5
10 Z=(Q-S)*HESJO(S*X)*HFSJO(S*Y)
RETURN
5 PRINT 9,P,S,F
9 FORMAT(4F10.5)
CALL EXIT
END

```

```

SUBROUTINE LAPINV(GLAM,PHI)
C THIS PROGRAM EVALUATES THE COEFFICIENTS FOR SERIES
C OF JACOBI POLYNOMIALS WHICH REPRESENTS A LAPLACE
C INVERSION INTEGRAL
REAL MUL
DIMENSION A(50),GLAM(50),PHI(50),C(4,50)
DIMENSION BK(101),TT(101)
COMMON/2/TI,TF,DT,MN,BK,TT
READ 1,MN,MN,MM
1 FORMAT(7I2)
READ 2,TI,TF,DT
2 FORMAT(3F10.5)
PRINT 99
99 FORMAT(1H1)
CALL SPLICE(GLAM,PHI,MM,C)
PRINT 101
101 FORMAT(/////5X,* GLAM PHI *)
PRINT 102,(GLAM(I),PHI(I),I=1,MM)
102 FORMAT(5X,F10.5,5X,F10.5)
M11=MN-1
PRINT 99
DO 10 I=1,MN
READ 3,BET,DEL
3 FORMAT(2F10.5)
PRINT 98,BET,DEL
98 FORMAT(/////5X,*BETA =*F5.3,* DELTA =*F5.3)
DO 11 L=1,MN
AL=L.
S=1./(AL+BET)/DEL
CALL SPLINE(GLAM,PHI,MM,C,S,G)
F=G*S
IF (AL-2.) 81,82,83
81 A(1)=(1.+BET)*DEL*F
GO TO 11
82 A(2)=((2.+BET)*DEL*F-A(1))*(3.+BET)
GO TO 11
83 CONTINUE
TOP=1.
L1=L-1
AL1=L1
DO 12 J=1,L1
AJ=J
TOP=AJ*TOP
12 CONTINUE
L2=2*L-1
ROT=1.
DO 13 J=L,L2
AJ=J
ROT=(AJ+BET)*ROT
13 CONTINUE
MUL=ROT/TOP
SUM=0.0
DO 14 N=1,L1
AN=N
IF (AN-2.) 85,86,87
85 TOP=1.
GO TO 88

```

ORIGINAL PAGE IS  
OF POOR QUALITY

```

A6 TOD=AL1
GO TO A9
A7 CONTINUE
TOD=1.
ICH=L1-(N-2)
DO 15 J=ICH,L1
AJ=J
TOD=AJ*TOD
15 CONTINUE
A8 CONTINUE
ROD=1.
JA=L1+N
DO 16 J=L,JA
AJ=J
ROD=ROD*(AJ+BET)
16 CONTINUE
CO=TOD/ROD
SUM=SUM+CO*A(N)
14 CONTINUE
A(L)=MUL*(DEL*F-SUM)
11 CONTINUE
CALL JACSER(DEL,A,BET)
10 CONTINUE
999 CONTINUE
RETURN
END

SUBROUTINE JACSER(D,C,H)
DIMENSION C(50),SF(50),P(50)
DIMENSION BK(101),TT(101)
COMMON/2/TT,TF,DT,MN,BK,TT
TT(1)=0.0
BK(1)=0.0
LM=1
T=TI
12 T=T+DT
X=2.*EXP(-D*T)-1.
CALL JACOBI(MN,X,B,P)
SF(1)=C(1)*P(1)
DO 10 L=2,MN
L1=L-1
AL=L
SF(L)=SF(L1)+C(L)*P(L)
10 CONTINUE
LM=LM+1
BK(LM)=SF(5)
TT(LM)=T
IF(T.LE.TF) GO TO 12
PRINT 97
97 FORMAT(/////5X,* T K T K T K
1 T K *)
DO 31 MY=1,25
MA=MY+1
MB=MA+25
MC=MB+25
MD=MC+25
PRINT 96,TT(MA),BK(MA),TT(MB),BK(MB),TT(MC),BK(MC),TT(MD),BK(MD)
96 FORMAT(5X,F5.2,3X,F7.5,3X,F5.2,3X,F7.5,3X,F5.2,3X,F7.5,3X,F5.2,3X,
1F7.5)
31 CONTINUE
RETURN
END

```



```

C      SUBROUTINE JACOB(N,X,R,PB)
C      THIS PROGRAM CALCULATES JACOBI POLYNOMIALS OF ORDER
      DIMENSION PB(N)
      AN=N
      IF(AN-2.)1,2,3
1     PB(1)=1.
      RETURN
2     PB(1)=1.
      PB(2)=X-B*(1.-X)/2.
      RETURN
3     RSQ=R*R
      RONE=B+1.
      PB(1)=1.
      PB(2)=X-R*(1.-X)/2.
      DO 4 K=3,N
        AK=K
        AK1=AK-1.
        AK2=AK-2.
        K1=K-1
        K2=K-2
        C01=((2.*AK1)+B)*X
        C01=((2.*AK2)+B)*C01
        C01=((2.*AK2)+BONE)*(C01-BSQ)
        C02=2.*AK2*(AK2+B)*((2.*AK1)+B)
        C0=2.*AK1*(AK1+R)*((2.*AK2)+R)
4     PB(K)=(C01*PB(K1)-C02*PB(K2))/C0
      RETURN
      END

```

ORIGINAL DATA  
OF POOR QUALITY

```

SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT)
DIMENSION X(50),Y(50),C(4,50)
IF(XINT-X(1))1,10,11
10 YINT=Y(1)
RETURN
11 CONTINUE
IF(X(M)-XINT)1,12,13
12 YINT=Y(M)
RETURN
13 CONTINUE
K=M/2
N=M
2 CONTINUE
IF(X(K)-XINT)3,14,5
14 YINT=Y(K)
RETURN
3 CONTINUE
IF(XINT-X(K+1))4,15,7
15 YINT=Y(K+1)
RETURN
4 CONTINUE
YINT=(X(K+1)-XINT)*(C(1,K)*(X(K+1)-XINT)**2+C(3,K))
YINT=YINT+(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
RETURN
5 CONTINUE
IF(X(K-1)-XINT)6,15,17
6 K=K-1
GO TO 4
16 YINT=Y(K-1)
RETURN
17 N=K
K=K/2
GO TO 2
7 LL=K
K=(N+K)/2
8 CONTINUE
IF(X(K)-XINT)3,14,18
18 CONTINUE
IF(X(K-1)-XINT)6,16,19
19 N=K
K=(LL+K)/2
GO TO 8
1 PRINT 101
101 FORMAT(* OUT OF RANGE FOR INTERPOLATION *)
STOP
END

```

```

SUBROUTINE SPLICE(Y,Y,M,C)
DIMENSION X(50),Y(50),D(50),P(50),E(50),C(4,50)
DIMENSION A(50,3),R(50),Z(50)
MM=M-1
DO 2 K=1,MM
D(K)=X(K+1)-X(K)
P(K)=D(K)/6.
2 E(K)=(Y(K+1)-Y(K))/D(K)
DO 3 K=2,MM
3 H(K)=E(K)-E(K-1)
A(1,2)=-1.-D(1)/D(2)
A(1,3)=D(1)/D(2)
A(2,3)=P(2)-P(1)*A(1,3)
A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
A(2,3)=A(2,3)/A(2,2)
R(2)=H(2)/A(2,2)
DO 4 K=3,MM
A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)
R(K)=H(K)-P(K-1)*R(K-1)
A(K,3)=P(K)/A(K,2)
4 B(K)=H(K)/A(K,2)
Q=D(M-2)/D(M-1)
A(M,1)=1.+Q*A(M-2,1)
A(M,2)=-Q-A(M,1)*A(M-1,3)
B(M)=B(M-2)-A(M,1)*B(M-1)
Z(M)=B(M)/A(M,2)
MN=M-2
DO 6 I=1,MN
K=M-I
6 Z(K)=H(K)-A(K,3)*Z(K+1)
Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
DO 7 K=1,MM
Q=1./(6.*D(K))
C(1,K)=Z(K)*Q
C(2,K)=Z(K+1)*Q
C(3,K)=Y(K)/D(K)-Z(K)*P(K)
7 C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
RETURN
END

```

Moment Intensity Factors

$$\mu_2/\mu_1 = 50.00 \quad \nu_1 = .30 \quad \nu_2 = .30$$

$$A/H = 1.00 \quad C_2/P_A = .02$$

.2500	.033865
.5000	.059829
.7500	.090175
1.0000	.257758

$$\mu_2/\mu_1 = 50.00 \quad \nu_1 = .30 \quad \nu_2 = .30$$

$$A/H = 1.00 \quad C_2/P_A = .04$$

.2500	.069941
.5000	.119396
.7500	.183013
1.0000	.352715

$$\mu_2/\mu_1 = 50.00 \quad \nu_1 = .30 \quad \nu_2 = .30$$

$$A/H = 1.00 \quad C_2/P_A = .06$$

.2500	.103272
.5000	.169658
.7500	.249217
1.0000	.412993

BETA = 0.000 DELTA = .200; ( $\rho_1 = \rho_2$ ;  $\nu_1 = \nu_2 = 0.3$ ;  $a/h = 1.0$ ;  $\mu_2/\mu_1 = 50.0$ )

T	K	T	K	T	K	T	K	T	K	GLAM	PHI
.10	.59591	2.60	.79662	5.10	.75682	7.60	.76022	0.00000	0.00000		
.20	.62915	2.70	.79458	5.20	.75628	7.70	.76082	.02000	.25776		
.30	.65817	2.80	.79247	5.30	.75582	7.80	.76142	.04000	.35271		
.40	.68339	2.90	.79031	5.40	.75543	7.90	.76205	.06000	.41299		
.50	.70522	3.00	.78813	5.50	.75511	8.00	.76268	.08000	.45535		
.60	.72402	3.10	.78595	5.60	.75486	8.10	.76333	.10000	.48761		
.70	.74010	3.20	.78379	5.70	.75468	8.20	.76399	.20000	.57966		
.80	.75378	3.30	.78167	5.80	.75457	8.30	.76465	.30000	.62537		
.90	.76530	3.40	.77959	5.90	.75451	8.40	.76533	.40000	.65346		
1.00	.77492	3.50	.77757	6.00	.75451	8.50	.76601	.50000	.67240		
1.10	.78286	3.60	.77562	6.10	.75457	8.60	.76669	.60000	.68644		
1.20	.78930	3.70	.77374	6.20	.75468	8.70	.76738	.70000	.69714		
1.30	.79443	3.80	.77194	6.30	.75484	8.80	.76808	.80000	.70558		
1.40	.79841	3.90	.77023	6.40	.75504	8.90	.76877	.90000	.71262		
1.50	.80139	4.00	.76860	6.50	.75529	9.00	.76947	1.00000	.71825		
1.60	.80348	4.10	.76707	6.60	.75558	9.10	.77017	2.00000	.74576		
1.70	.80482	4.20	.76563	6.70	.75591	9.20	.77087	3.00000	.75511		
1.80	.80549	4.30	.76428	6.80	.75628	9.30	.77157	4.00000	.76103		
1.90	.80561	4.40	.76303	6.90	.75667	9.40	.77227	5.00000	.76423		
2.00	.80524	4.50	.76187	7.00	.75710	9.50	.77296				
2.10	.80447	4.60	.76081	7.10	.75756	9.60	.77366				
2.20	.80336	4.70	.75983	7.20	.75805	9.70	.77435				
2.30	.80197	4.80	.75895	7.30	.75856	9.80	.77503				
2.40	.80036	4.90	.75816	7.40	.75909	9.90	.77572				
2.50	.79856	5.00	.75745	7.50	.75965	10.00	.77640				

Derivation of Equation (25). Equation (25) can be derived by first expressing equation (24) in the form

$$\begin{aligned} T + V = & \int_{t_0}^t dt \iint \left( \frac{\rho_0 h^3}{24} \left[ \frac{\partial \psi_x}{\partial t} \frac{\partial^2 \psi_x}{\partial t^2} + \frac{\partial \psi_y}{\partial t} \frac{\partial^2 \psi_y}{\partial t^2} \right] + \frac{\bar{\rho} h}{2} \frac{\partial w}{\partial t} \frac{\partial^2 w}{\partial t^2} \right) dx dy \\ & + \int_{t_0}^t dt \iint \frac{\partial \bar{W}}{\partial t} dx dy + \bar{T}_0 + \bar{V}_0 \end{aligned} \quad (59)$$

in which  $\partial \bar{W} / \partial t$  can be written as

$$\begin{aligned} \frac{\partial \bar{W}}{\partial t} = & \frac{\partial \bar{W}}{\partial \Gamma_x} \frac{\partial \Gamma_x}{\partial t} + \frac{\partial \bar{W}}{\partial \Gamma_y} \frac{\partial \Gamma_y}{\partial t} + \dots + \frac{\partial \bar{W}}{\partial \Gamma_{yz}} \frac{\partial \Gamma_{yz}}{\partial t} \\ = & (M_x \frac{\partial}{\partial x} + H_{xy} \frac{\partial}{\partial y} + Q_x) \frac{\partial \psi_x}{\partial t} \\ & + (H_{xy} \frac{\partial}{\partial x} + M_y \frac{\partial}{\partial y} + Q_y) \frac{\partial \psi_y}{\partial t} \\ & + (Q_x \frac{\partial}{\partial x} + Q_y \frac{\partial}{\partial y}) \frac{\partial w}{\partial t} \end{aligned} \quad (60)$$

Denoting  $n$  and  $s$  as the normal and tangential direction, equation (60) may be integrated to yield

$$\begin{aligned} \iint \frac{\partial \bar{W}}{\partial t} dx dy = & \oint \left( \frac{\partial \psi_n}{\partial t} M_n + \frac{\partial \psi_s}{\partial t} H_{ns} + \frac{\partial w}{\partial t} Q_n \right) ds - \iint \left[ \frac{\partial \psi_x}{\partial t} \left( \frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - Q_x \right) \right. \\ & \left. + \frac{\partial \psi_y}{\partial t} \left( \frac{\partial H_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y \right) + \frac{\partial w}{\partial t} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) \right] dx dy \end{aligned} \quad (61)$$

Putting equation (61) into (59) and observing the relations in equations (15), the expression for  $\bar{T} + \bar{V}$  in equation (25) is obtained.